Recursive Estimation of ARX Systems Using Binary Sensors with Adjustable Thresholds

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Outline

• Problem: identifying an ARX systems via binary sensors

• Previous solutions typically assumed fully known noise characteristics

• They also assumed that the input signal can be chosen by the user

• We try to reduce the assumptions on the noise and the input

• Full knowledge on the distribution is not needed; the input is only observed

• But, the threshold of the binary sensor can be controlled ～ dither signal

• Here, two recursive identification algorithms are proposed

• Algorithm I: FIR approximation; it is proved to be strongly consistent

• Algorithm II: simultaneous state and parameter estimation (simulations)
Structural Overview

**Part I.** Problem Setting
(ARX System via Binary Sensors, Dithering, Assumptions)

**Part II.** General Form of the Algorithms
(Sign-Error, Step-Sizes, Expanding Truncation Bounds)

**Part III.** Recursive Identification: Algorithms I and II
(FIR Approximation, Strong Consistency, Simultaneous Estimation)

**Part IV.** Experimental Results
(Simulation: Algorithms I and II on an ARX(2,2) System)

**Part V.** Summary and Concluding Remarks
(Main Ideas, Contributions and Highlights)
Problem Setting

• We observe an ARX system via a binary sensor:

\[ X_t \triangleq \sum_{i=1}^{p} a_i^* X_{t-i} + \sum_{i=1}^{q} b_i^* U_{t-i} + N_t, \]

\[ Y_t \triangleq \mathbb{I}(X_t \leq C_t), \]

where \( X_t \) — output (hidden state), \( U_t \) — input, \( N_t \) — noise (at time \( t \))

• The thresholds of the binary sensor, \( (C_t)_t \), can be controlled at each \( t \)

• Data: the inputs \( (U_t)_t \) and the binary outputs \( (Y_t)_t \) are observed

• Aim: to identify (estimate) \( \theta^* = (a_1^*, \ldots, a_p^*, b_1^*, \ldots, b_q^*) \in \mathbb{R}^{p+q} \)
Adjustable Thresholds $\sim$ Dithering

• The binary output can be rewritten as

$$Y_t = \mathbb{I}(\varphi_t^T \theta^* + N_t \leq C_t) = \mathbb{I}(\varphi_t^T \theta^* + N_t - C_t \leq 0),$$

where $\varphi_t = (X_{t-1}, \ldots, X_{t-p}, U_{t-1}, \ldots, U_{t-q})$ — random regressor

• Choosing the threshold is equivalent to dithering
System Assumptions

• \((N_t)_t\) is i.i.d., continuous, zero mean, zero median, has a finite variance:
  \[ \sigma_n^2 \triangleq \mathbb{E} [N_t^2] < \infty, \]
  and has a continuous and positive density at zero.

• \((U_t)_t\) is i.i.d., zero mean, \((U_t)_t\) and \((N_t)_t\) are independent, and
  \[ 0 < \sigma_u^2 < \infty, \]
  where \(\sigma_u^2 \triangleq \mathbb{E} [U_t^2]\).

• The system is stable, i.e., the roots of \(A^*(z)\) lie strictly inside the unit circle; additionally, the transfer function \(B^*(z)/A^*(z)\) is irreducible,

\[
A^*(z) \triangleq 1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_p z^{-p},
\]
\[
B^*(z) \triangleq b_1 z^{-1} + b_2 z^{-2} + \cdots + b_q z^{-q},
\]

where \(z^{-1}\) is the backward shift operator, \(z^{-i} x_t \triangleq x_{t-i}\).

• The orders \(p\) and \(q\) are known.
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**General Form of the Algorithms**

- The general form of both proposed algorithms is
  \[
  \hat{\theta}_{t+1} = \Pi_{M_{\mu(t)}} \left[ \hat{\theta}_t + \alpha_t \hat{\varphi}_t \left( 1 - 2 \mathbb{I}(X_t \leq \hat{\varphi}_T^T \hat{\theta}_t) \right) \right],
  \]
  where \( \hat{\varphi}_t \) is a regression vector defined differently in the two algorithms, \( (\alpha_t)_t \) is a sequence of step-sizes and \( \Pi_{M_{\mu(t)}} \) is a sequence of projections.

- Assuming that \( N_t \) is continuous, we (\( \mathbb{P} \)-a.s.) have
  \[
  \text{sign}(X_t - \hat{\varphi}_T^T \hat{\theta}_t) = 1 - 2 \mathbb{I}(X_t \leq \hat{\varphi}_T^T \hat{\theta}_t),
  \]
  which is a sign-error type algorithm with expanding truncation bounds.
Step-Sizes

- Typical step-size assumption of stochastic approximation algorithms

\[
\sum_{t=0}^{\infty} \alpha_t = \infty, \\
\sum_{t=0}^{\infty} \alpha_t^2 < \infty,
\]

\(\forall t \geq 0 : \alpha_t \geq 0.\)

The second condition can often be weakened to \(\lim_{t \to \infty} \alpha_t = 0\)

- Here, we will simply assume that

\[\alpha_0 = 1 \quad \text{and} \quad \forall t > 0 : \alpha_t = 1/t.\]
Expanding Truncation Bounds

- Let $(M_t)_t$ be a sequence of (strictly) monotone increasing positive real numbers with $M_t \to \infty$ as $t \to \infty$.

- Let $\mathbb{I}(\cdot)$ be the indicator function and define $\mu(t)$ and $\Delta \hat{\theta}_i$ as

$$
\mu(t) \triangleq \sum_{i=1}^{t-1} \mathbb{I}(|\hat{\theta}_i + \Delta \hat{\theta}_i| > M_{\mu(i)}),
$$

$$
\Delta \hat{\theta}_i \triangleq \alpha_i \varphi_i (1 - 2 \mathbb{I}(X_i \leq \varphi_i^T \hat{\theta})).
$$

- Given a positive real $M$, projection $\Pi_M$ is

$$
\Pi_M(x) \triangleq \begin{cases} 
  x & \text{if } \|x\| \leq M, \\
  0 & \text{otherwise}.
\end{cases}
$$
Algorithm I: FIR Approximation

- Using impulse responses, \((c_i^*)_{i=1}^{\infty}\) and \((d_i^*)_{i=0}^{\infty}\), we have

\[
X_t = \sum_{i=1}^{\infty} c_i^* U_{t-1} + \sum_{i=0}^{\infty} d_i^* N_{t-i},
\]

- Let’s approximate our ARX system with an FIR system of order \(p + q\)

\[
X_t = \bar{\varphi}_t^T \bar{\theta}^* + W_t,
\]

\[
\bar{\varphi}_t \triangleq (U_{t-1}, \ldots, U_{t-p-q})^T, \quad \bar{\theta}^* \triangleq (c_1^*, \ldots, c_{p+q}^*)^T.
\]

- \(W_t\) is simply the unmodelled part of the system

\[
W_t \triangleq \sum_{i=p+q+1}^{\infty} c_i^* U_{t-i} + \sum_{i=0}^{\infty} d_i^* N_{t-i}.
\]
Algorithm I: FIR Approximation

- If we can estimate $\bar{\theta}^*$, we can also estimate the true parameter vector $\theta^*$
- There is a function $f$, which we use for post processing, such that
  \[ \theta^* = f(\bar{\theta}^*) , \]
- Algorithm I is defined by using $\hat{\varphi}_t \triangleq \varphi_t$ in the General Algorithm

**Theorem 1 (Strong Consistency of Algorithm I).** Let $(\hat{\theta}_t)_{t=0}^{\infty}$ be the sequence generated by Algorithm I (i.e. $\hat{\varphi}_t = \varphi_t$). Then, under the given assumptions, $f(\hat{\theta}_t)$ converges (\(\mathbb{P}\)-a.s.) to $\theta^*$, as $t \to \infty$, for any $\hat{\theta}_0 \in \mathbb{R}^{p+q}$.

- Furthermore, $\sqrt{t}(\hat{\theta}_t - \bar{\theta}^*)$ is approximately normal
Algorithm II: Simultaneous Estimation

- **Main idea:** to achieve a direct estimate of $\theta^*$ by simultaneously maintaining an estimate for the output, $\hat{X}_t$ and for the parameter, $\hat{\theta}_t$, at time $t$.

- **The sequence of output estimates** is defined as

$$
\hat{X}_t \triangleq \begin{cases} 
\sum_{i=1}^{p} \hat{a}_{t,i} \hat{X}_{t-1} + \sum_{i=1}^{q} \hat{b}_{t,i} U_{t-i} & \text{if } t \geq 0 \\
0 & \text{otherwise,}
\end{cases}
$$

where $(\hat{a}_{t,i})_{i=1}^{p}$ and $(\hat{b}_{t,i})_{i=1}^{q}$ are the estimates of the true parameters.

- **Algorithm II:** is defined by setting the General Algorithm as

$$
\hat{\varphi}_t \triangleq (\hat{X}_{t-1}, \ldots, \hat{X}_{t-p}, U_{t-1}, \ldots, U_{t-q})^T,
$$

$$
\hat{\theta}_t \triangleq (\hat{a}_{t,1}, \ldots, \hat{a}_{t,p}, \hat{b}_{t,1}, \ldots, \hat{b}_{t,q})^T.
$$
Simulation Experiment: ARX(2, 2)

Figure 1: Recursive estimation with Algorithm I
Simulation Experiment: ARX(2, 2)

Figure 2: Recursive estimation with Algorithm II
Summary and Concluding Remarks

- Two recursive identification algorithms have been proposed for identifying ARX systems via binary sensors.
- These algorithms neither assume the knowledge of the particular noise distributions, nor assume that the input signal can be chosen by the user.
- But, they do assume that the threshold of the sensor can be controlled.
- This assumption is equivalent to allowing a dither signal.
- Algorithm I: FIR approximation; it was proved to be strongly consistent.
- Algorithm II: simultaneous state and parameter estimation (no theorem).
- Experimental results demonstrated that both algorithms efficiently approximated the parameters of an ARX(2,2) system.
Thank you for your attention!

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