Score Permutation Based Finite Sample Inference for GARCH Models

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Overview
- Confidence regions (and hypothesis testing) for parameters of GARCH processes
- ScoPe: works by permuting the residuals in the score (gradient of the log-likelihood)
- Centered around the Quasi-Maximum Likelihood Estimate (QMLE)
- Distribution-free (w.r.t. the driving noise; even if it is heavy-tailed and skewed)
- Non-asymptotic (finite sample) guarantees
- Exact (user-chosen) coverage probabilities
- Applicable to nonstationary models, as well
- Confirmation on major stock market indices

Asymptotics of QMLE
Under mild regularity conditions (nondeg, noise & identity), the QMLE is strongly consistent
\[ \hat{\theta}_{\text{QMLE}} \to \theta^* \quad \text{as} \quad n \to \infty. \]
It can also be proved, assuming \( E[\varepsilon_t^4] < \infty \), that the QMLE is asymptotically normal
\[ \sqrt{n}(\hat{\theta}_n - \theta^*) \to \mathcal{N}(0, \Gamma) \quad \text{as} \quad n \to \infty, \]
for a covariance matrix \( \Gamma \) depending on \( \nabla \theta^2(\theta^*) \). This can be used to define (asymptotic) confidence ellipsoids. Assume \( \Gamma_{\theta}(s) \) is an estimate of \( \Gamma \), then
\[ \hat{\Theta}_n(s) = \left\{ \theta \in \mathbb{R}^{1+q} : \theta - \hat{\theta}_n \right\} \Gamma_{\hat{\Theta}_n}(s) \leq \frac{8}{n} \}, \]
where \( d = p + q + 1 \) and the probability that \( \theta^* \in \hat{\Theta}_n \) is approximately \( F_{\chi^2}(d) \), which is the CDF of the \( \chi^2 \) distribution with \( d \) degrees of freedom.

Gaussian Score
The QMLE satisfies the likelihood equation
\[ \nabla \ell_n(\hat{\theta}_n) = 0, \]
and the gradient of the (conditional) log-likelihood function, the score function, can be written as
\[ \nabla \ell_n(\theta) = 2 \sum_{i=1}^n (1 - \varepsilon_i^2(\theta)) \frac{1}{\sigma_i^2(\theta)} \nabla \sigma_i^2(\theta), \]
where \( \varepsilon_i(\theta) = X_i/\sigma_i(\theta) \) is a reconstructed residual for time \( t \) assuming parameter \( \theta \), and \( \sigma_i(\theta) \) is an estimate of \( \sigma_i \), which can be calculated recursively.

Score Permutation
Note that \( \varepsilon_i(\theta^*) = \varepsilon_i \) for all \( t \), assuming
(P1) The “true” system is in the model class.
(P2) The initial conditions are known.
Since \( \{\varepsilon_i\} \) is i.i.d., their joint distribution is maintained under arbitrary index permutation \( \pi(\cdot) \).
\[ \{\varepsilon_i(\cdot)\} \equiv \{\varepsilon_{\pi(\cdot)}\}, \]
for all \( i \in \{1, \ldots, m-1\} \), where \( m \) is user-chosen. Let \( \pi_0 \) be the identity permutation, i.e., \( \pi_0(t) = t \). The original and the perturbed score functions are
\[ B(\theta, \pi_i) \equiv \frac{1}{n} \sum_{i=1}^n \frac{(1 - \varepsilon_i^2(\theta))}{\sigma_i^2(\theta, \pi_i)} \nabla \sigma_i^2(\theta, \pi_i), \]
where the perturbed variances \( \sigma_i^2(\theta, \pi_i) \) are
\[ \sigma_i^2(\theta, \pi_i) \equiv \omega + \sum_{j=1}^p \alpha_j X_{i-j}^2 + \sum_{j=1}^q \beta_j \sigma_{i-j}^2(\theta), \]
which gives rise to an alternative trajectory
\[ X_i(\theta, \pi_i) \equiv \hat{\sigma}_i^2(\theta, \pi_i) \varepsilon_i(\cdot), \]
The rank of \( \| B(\theta, \pi_0) \|^2 \) within \( \{\| B(\theta, \pi_i) \|^2 \} \) is
\[ R(\pi_i) \equiv \frac{1}{n} \sum_{i=1}^n \| (B(\theta, \pi_i) - B(\theta, \pi_0)) \|^2, \]
where \( \| \cdot \| \) is an indicator function and \( > \) is > with random tie-breaking. The ScoPe confidence set is
\[ \hat{\Theta}_n(m, r) \equiv \left\{ \theta \in \Theta : R(\pi_i) \leq m - r \right\}, \]
where \( m > r > 0 \) are user-chosen integers.

Experimental Results
The experiments focused on GARCH(1,1) models
\[ X_t \equiv \sigma_t \varepsilon_t, \]
\[ \sigma_t^2 \equiv \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2, \]
using both simulated and real-world datasets. ScoPe was compared with asymptotic ellipsoids, residual- and likelihood ratio bootstrap methods.
The daily closing prices of Nasdaq 100, S&P 500 and FTSE 100 were used from the entire period of 2014. Models were fitted to the compounded returns, i.e., for each price sequence \( \{P_t\} \), the data were transformed by \( R_t = \log(P_t/P_{t-1}) \).

Table 1: Relative Areas on Stock Market Indices (2014)

<table>
<thead>
<tr>
<th>Method</th>
<th>Nasdaq 100</th>
<th>S&amp;P 500</th>
<th>FTSE 100</th>
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</thead>
<tbody>
<tr>
<td>Asym. Ell.</td>
<td>0.3426</td>
<td>0.1679</td>
<td>0.1535</td>
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<tr>
<td>Res. Boots.</td>
<td>0.3791</td>
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<td>0.2850</td>
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<tr>
<td>LR. Boots.</td>
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<td>ScoPe</td>
<td>0.3601</td>
<td>0.2862</td>
<td>0.2412</td>
</tr>
</tbody>
</table>

Figure 1: Logistic noise, \( n = 100, m = 100, r = 10 \); Exact 90% ScoPe confidence set for a stationary GARCH(1,1).

References


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